ON SOME PROBLEMS WITH UNKNOWN BOUNDARIES FOR THE HEAT CONDUCTION EQUATION

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A certain class of problems with unknown boundaries are considered herein in connection with the problem, posed by Barenblatt and Ishlinskii [1], on the impact of a viscoplastic rod on a rigid obstacle, which was the fundamental model for typification of this class. The presence of singularities in the unknown functions (the desired solution of the heat conduction equation has a discontinuous point, the derivatives of the unknown boundary are unbounded), and the nonmonotonous behavior of the unknown boundary are characteristic of the considered problems (*).

A theorem on the uniqueness of the solution of these problems is established, functional equations are derived for the unknown boundaries (equivalent to an initial value problem), and some properties of the solution are discussed (in more detail in the case of the above-mentioned problem of impact of a rod).

1. Formulation of the problems. It is required to find a continuous function $h(t)$, $h(0) = 0$, $h(t) > 0$ for $t \in (0, T)$ in some segment $0 \leqslant t \leqslant T$, and a bounded solution of the heat conduction equation

$$u_t = u_{xx} \tag{1.1}$$

in the domain

$$\Omega = \{(t, x) : 0 < x < h(t),\ 0 < t \leqslant T\}$$

which is continuous in $\overline{\Omega} \setminus (0,0)$ together with the derivative $u_x(t, x)$ and satisfies the conditions

$$u\,|_{x=0} = f(t), \quad u_x\,|_{x=h(t)} = 0, \quad u\,|_{x=h(t)} = A(h) \equiv g^h(t) \tag{1.2}$$

where $f(t)$ is a continuous function for $t \geqslant 0$, and A is some operator with values in the space of continuously differentiable functions $C^1[0, T]$. From the assumption on the smoothness of the function $g(t)$, in some cases the smoothness of the function $h(t)$ naturally results(**): for example, if $g(t) = F(h(t))$, where $F(\sigma)$ is a smooth monotone function; moreover, some constraints on $h(t)$ may derive from the very formulation of the problem (in particular, from physical considerations; for example, in the case of the problem of impact of a rod $h(t)$ is less than the rod length). Hence, the operator A is generally defined only on some subset of continuous functions which we shall designate the set of admissible boundaries, and whose selection is essential in the formulation of the problem.

Let T^* be the upper bound of those T for which a unique solution of the considered problem exists; if $T^* < +\infty$ (for example, $h(T) = 0$), it is then said that the solution exists, locally, in the small.

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* Problems with monotonous unknown boundaries, having regular solutions, are investigated in [2 and 3], say.

** We shall not henceforth stress the dependence of the function $g(t)$ on $h(t)$ in those cases where this may certainly not lead to misunderstanding.

2. Representation of the solution $u(t, x)$ which isolates its principle singularity. When $f(0) \neq g(0)$, the function $u(t, x)$ is known to be discontinuous at the point (0, 0). Hence, let us first establish a representation for the function $u(t, x)$ which is not connected with the specific form of the operator A but which isolates its discontinuous part.

Theorem 2.1. Let $h(t)$, $u(t, x)$ form a possible solution of the problem (1.1), (1.2). Then

$$u(t, x) = g(0)\Phi\left(\frac{x}{2\sqrt{t}}\right) + \frac{1}{2\sqrt{\pi}}\int_0^t \frac{x}{(t-\tau)^{3/2}}\exp\left(-\frac{x^2}{4(t-\tau)}\right) f(\tau)\,d\tau +$$
$$+\frac{1}{2}\int_0^t \left[\Phi\left(\frac{x+h(\tau)}{2\sqrt{t-\tau}}\right) + \Phi\left(\frac{x-h(\tau)}{2\sqrt{t-\tau}}\right)\right] g'(\tau)\,d\tau \qquad (2.1)$$
$$\left(\Phi(\sigma) = \frac{2}{\sqrt{\pi}}\int_0^\sigma \exp(-\sigma^2)\,d\sigma\right)$$

Proof. Let us predetermine the function $u(t, x)$ in the domain $\{(t, x) : h(t) < x < \infty,\ 0 \leqslant t \leqslant T\}$ by means of the formula $u(t, x) \equiv g(t)$. The function $u(t, x)$ obtained on the half-axis $\pi\ \{(t, x) : 0 \leqslant x < +\infty,\ 0 \leqslant t \leqslant T\}$ is continuous in $\pi \setminus (0,0)$ together with its derivative $u_x(t, x)$ and is a bounded solution of the problem:

$$u_t - u_{xx} = F(t, x) \equiv \begin{cases} 0 & \text{for } x < h(t) \\ g'(t) & \text{for } x > h(t) \end{cases} \qquad (2.2)$$
$$u|_{x=0} = f(t),\quad u|_{t=0} = g(0),\quad [u]|_{x=h(t)} = [u_x]|_{x=h(t)} = 0 \qquad (2.3)$$

The solution of such a problem may be expressed in terms of the Green's function

$$G(x, \xi, t, \tau) = \frac{1}{2\sqrt{\pi(t-\tau)}}\left[\exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) - \exp\left(-\frac{(x+\xi)^2}{4(t-\tau)}\right)\right]$$

by the following formula(*):

$$u(t, x) = \int_0^{+\infty} g(0)\, G(x, \xi, t, 0)\,d\xi + \int_0^t G_\xi(x, 0, t, \tau) f(\tau)\,d\tau +$$
$$+\int_0^t\int_0^{+\infty} G(x, \xi, t, \tau) F(\tau, \xi)\,d\xi\,d\tau \qquad (2.4)$$

from which we obtain the representation (2.1) by elementary manipulation.

Corollary 2.1. The function

$$u(t, x) - g(0)\Phi\left(\frac{x}{2\sqrt{t}}\right) - f(0)\left(1 - \Phi\left(\frac{x}{2\sqrt{t}}\right)\right)$$

is continuous in $\overline{\Omega}$, where (the identity $f(\tau) = f(\tau) - f(0) + f(0)$) is applied to transform the second member of the formula):

$$\left|u(t, x) - g(0)\Phi\left(\frac{x}{2\sqrt{t}}\right) - f(0)\left(1 - \Phi\left(\frac{x}{2\sqrt{t}}\right)\right)\right| \leqslant$$
$$\leqslant \max_{0\leqslant\tau\leqslant t} |f(\tau) - f(0)| + t \max_{0\leqslant t\leqslant T} |g'(t)|$$

from which for $t \to 0$

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*) The proof of (2.4) may be obtained by direct verification that the function $u(t, x)$ in (2.4) (or in (2.1)) yields the solution of the problem (2.2), (2.3), and by application of the uniqueness theorem for the solution of the problem (2.2), (2.3) in the class of bounded functions.

$$\left|(g(0)-f(0))\left(1-\Phi\left(\frac{h(t)}{2\sqrt{t}}\right)\right)\right| = o(1) \leqslant$$
$$\leqslant \max_{0\leqslant\tau\leqslant t} |f(\tau)-f(0)| + 2t \max_{0\leqslant t\leqslant T} |g'(t)|$$

Corollary 2.2. If $g(0) - f(0) \neq 0$, then

$$\lim_{t\to 0} \frac{h(t)}{\sqrt{t}} = +\infty \tag{2.5}$$

which follows from the second estimate of corollary 2.1.

Corollary 2.3. (Maximum principle). The inequalities

$$\min [g(0), \min_{0\leqslant\tau\leqslant t} f(\tau)] \leqslant u(t, x) \leqslant \max [g(0), \max_{0\leqslant\tau\leqslant t} f(\tau)]$$

are valid for the function $u(t, x)$ (it is sufficient to apply the customary maximum principle for $t \geqslant \delta > 0$, and to let δ tend to zero).

Corollary 2.4. For $f(t) \equiv \text{const} = f(0)$ the problem (1.1), (1.2) has meaning only under compliance with the condition

$$g(0) - f(0) \neq 0$$

Indeed, we otherwise obtain from (2.1) that the function $u(t, x)$ is continuous at the origin, and therefore, equals the constant $f(0)$ for any boundary $x = h(t)$.

Since we are especially interested in the case of the discontinuous function $u(t, x)$, and the case $f(t) \equiv 0$ (in the problem of rod impact), then we shall henceforth consider this condition to be satisfied everywhere.

3. Uniqueness theorem. Let us assume that the operator A in the set of admissible boundaries has the following properties.

1) The value of $A(h)|_{t=0} = g(0)$ is independent of the selection of $h(t)$.

2) If $h_1(t_0) > h_2(t_0)$, then a point $t^* < t_0$ exists such that $h_1(t^*) = h_2(t^*)$ and for $g^{h_i}(t) \equiv A(h_i)$ $(i = 1, 2)$, the inequality

$$g^{h_1}(t^*) - g^{h_1}(t_0) \geqslant g^{h_2}(t^*) - g^{h_2}(t_0) \tag{3.1}$$

is valid.

For example, if $A(h) \equiv F(h(t))$, where $F(\sigma)$ is a smooth decreasing function, then the property (2) is satisfied for any t_0 with $t^* = 0$. As will be shown below, the problem of impact of a viscoplastic rod yields another example of such an operator.

With respect to the smooth function $f(t)$ we assume that(*)

$$f(0) \leqslant g(0), \qquad f'(t) \leqslant 0 \tag{3.2}$$

Let us note that in the case of the smooth function $f(t)$ the second member on the right-hand side of (2.1) (we denote it by J_f) transforms after integration by parts into

$$J_f = f(t) - f(0)\Phi\left(\frac{x}{2\sqrt{t}}\right) - \int_0^t \Phi\left(\frac{x}{2\sqrt{\tau}}\right) f'(t-\tau)\, d\tau \tag{3.3}$$

Theorem 3.1. Under the mentioned assumptions relative to the operator A and the function $f(t)$ the solution of the problem (1.1), (1.2) is unique.

Proof. First, we obtain by using the representation (2.1) for any solution $h(t)$, $u(t, x)$ of the problem (1.1), (1.2) which takes account of (3.2) and (3.3), and by applying the maximum principle in the domain $\{(t, x)\colon 0 \leqslant x \leqslant h(t),\ 0 < \delta \leqslant t \leqslant t_0 \leqslant T\}$, that $u(t_0, x) \geqslant f(t_0) - c_1\delta$, from which in the limits as $\delta \to 0$, there results that $u(t, x) \geqslant f(t)$ in the domain Ω and therefore, $u_x(t, 0) \geqslant 0$.

Differentiating (2.1) with respect to x we find that

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*) The case $f(0) \geqslant g(0)$, $f'(t) \geqslant 0$ reduces to (3.2) by multiplying the "data" of problem (1.1), (1.2) by -1.

$$u_x(t,\ x) \geqslant \frac{1}{2}\int_0^t \frac{1}{\sqrt{\pi(t-\tau)}}\left[\exp\left(-\frac{(x+h(\tau))^2}{4(t-\tau)}\right)+\exp\left(-\frac{(x-h(\tau))^2}{4(t-\tau)}\right)\right]g'(\tau)\,d\tau$$

from which the inequality $u_x(\delta,\ x) \geqslant -c_2\delta^{1/2}$ follows. According to the maximum principle for a function $u_x\ (t,\ x)$ continuous in the domain $\{(t,\ x)\colon 0 \leqslant x \leqslant h(t),\ 0 < \delta \leqslant t \leqslant T\}$ which satisfies (1.1), we obtain by taking account of the second condition in (1.2) that $u_x(t,\ x) \geqslant -c_2\delta^{1/2}$ for $t \geqslant \delta$; letting δ tend to 0, we find that in $\overline{\Omega}\setminus(0,\ 0)$

$$u_x(t,\ x) \geqslant 0 \tag{3.4}$$

Hence, because of the strengthened maximum principle [4],

$$u_x(t,\ x) > 0 \tag{3.5}$$

at points of the domain Ω.

Now, let us assume that two solutions exist for the problem (1.1), (1.2): $h_1(t)$, $u_1(t,\ x)$ on the segment $[0,\ T_1]$ and $h_2(t)$, $u_2(t,\ x)$ on the segment $[0,\ T_2]$. Let

$T = \min(T_1,\ T_2)$, $m(t) = \min[h_1(t),\ h_2(t)]$ and $D = \{(t,\ x)\colon 0 < x < m(t),\ 0 < t \leqslant T\}$

Let us consider the function $w(t,\ x) \equiv u_1(t,\ x) - u_2(t,\ x)$, $w(0,0) = 0$, in the domain $\overline{D}$, which satisfies (1.1) in D. By virtue of Theorem 2.1 this function is continuous in $\overline{D}$ and $w(t,\ x) \to 0$ as $t \to 0$.

Let P be the point of the maximum of the function $w(t,\ x)$ in $\bar{\bar{D}}$, where $w(P) > 0$. Since $w(t,\ 0) = 0$, the point P should lie on the curve $x = m(t)$, i.e., $P = (t_0,\ m(t_0))$, $t_0 \in (0,\ T]$. According to the known property of the solutions of the heat conduction Eqs. $w_x(P) > 0$ (let us note that $w \neq \text{const}$). Therefore, $h_1(t_0) > h_2(t_0)$, because otherwise, by virtue of the second condition in (1.2) for the function u_1 and the inequalities (3.4) for the function u_2:

$$w_x(P) = u_{1x}(P) - u_{2x}(P) = -u_{2x}(P) \leqslant 0$$

According to the property (2) of the operator A a point $t^* < t_0$ exists for which $h_1(t^*) = h_2(t^*)$ and inequality (3.1) is satisfied. Applying the inequality (3.5) for the function $u_1(t,\ x)$ and the inequality (3.1) successively, we have

$$w(P) = u_1(t_0,\ m(t_0)) - g^{h_2}(t_0) < u_1(t_0,\ h_1(t_0)) - g^{h_2}(t_0) = g^{h_1}(t_0) - g^{h_2}(t_0) \leqslant$$
$$\leqslant g^{h_1}(t^*) - g^{h_2}(t^*) = w(t^*,\ m(t^*))$$

But this contradicts the selection of the point P as the maximum point of the function $w(t,\ x)$ in $\overline{D}$. Therefore $w(P) \leqslant 0$, which means that everywhere in $\overline{D}$:

$$u_1(t,\ x) \leqslant u_2(t,\ x)$$

Since the functions u_1 and u_2 may be interchanged from the very beginning of the proof, the reverse inequality is also valid. Hence, $u_1 \equiv u_2$ in $\overline{D}$.

Let us show that then $h_1(t) \equiv h_2(t)$ also in $[0,\ T]$. In fact, if the point $\theta \in [0,\ T]$ were such that $h_1(\theta) < h_2(\theta)$ say, then according to the second condition in (1.2), $u_{1x}(\theta,\ h_1(\theta)) = 0$, and according to (3.5) $u_{1x}(\theta,\ h_1(\theta)) = u_{2x}(\theta,\ h_1(\theta)) > 0$; we have arrived at a contradiction. Theorem 3.1 is proved completely.

Remark 3.1. We note that in the proof of the theorem of uniqueness none of the smoothness of unknown boundaries has been utilized.

4. Reduction of the problem (1.1), (1.2) to equivalent functional equations for an unknown boundary. Let $h(t)$, $u(t,\ x)$ be the solution of problem (1.1), (1.2). Let us represent the function $u(t,\ x)$ by means of (2.1) and let x tend to $h(t)$ therein for fixed t (the foundation for the passage to the limit under the integral is obvious here). Utilizing the third condition in (1.2), we find

$$g(0)\left[1-\Phi\left(\frac{h(t)}{2\sqrt{t}}\right)\right] = \frac{1}{2\sqrt{\pi}}\int_0^t \frac{h(t)}{(t-\tau)^{3/2}}\exp\left(-\frac{h^2(t)}{4(t-\tau)}\right)f(\tau)\,d\tau -$$
$$-\frac{1}{2}\int_0^t\left[2-\Phi\left(\frac{h(t)+h(\tau)}{2\sqrt{t-\tau}}\right)-\Phi\left(\frac{h(t)-h(\tau)}{2\sqrt{t-\tau}}\right)\right]g'(\tau)\,d\tau \tag{4.1}$$

Conversely, let $h(t)$ be a continuous solution of (4.1) for which $h'(t)$ exists in the half-interval $(0, T]$ and

$$\int_0 |h'(\tau)|(t-\tau)^{-1/2}\,d\tau < \infty$$

Let us show that the function $u(t, x)$, defined by (2.1), satisfies all the requirements of the definition of the solution of the problem (1.1), (1.2). Not evident is only the condition $u_x|_{x=h(t)} = 0$; let us verify it. Let us note that the considered function $u(t, x)$ has continuous derivatives u_x, u_t, u_{xx} in $\overline{\Omega}\setminus(0,0)$ and let us apply the integral representation for $u(t, x)$ by using the function $G(x, \xi, t, \tau)$ mentioned in Section 1 (see [5], Ch. VI, Sec. 3) while taking into account that $|u_x(t, x)| \leqslant c_3/t^{1/2}$:

$$u(t, x) = \int_0^t G_\xi(x, 0, t, \tau) f(\tau)\,d\tau - \int_0^t G(x, h(\tau), t, \tau)\,u_x(\tau, h(\tau))\,d\tau +$$

$$+ \int_0^t [G(x, h(\tau), t, \tau)\,h'(\tau) - G_\xi(x, h(\tau), t, \tau)]\,g(\tau)\,d\tau$$

Using the fact that the square brackets in the last member equals

$$-\frac{1}{2}\frac{d}{d\tau}\left[\Phi\left(\frac{x+h(\tau)}{2\sqrt{t-\tau}}\right) + \Phi\left(\frac{x-h(\tau)}{2\sqrt{t-\tau}}\right)\right]$$

and transforming this member by integration by parts, we obtain a representation for $u(t, x)$ which differs from (2.1) only by the term

$$V(t, x) \equiv \int_0^t G(x, h(\tau), t, \tau)\,u_x(\tau, h(\tau))\,d\tau$$

which therefore equals zero in $\overline{\Omega}$. Utilizing the known property of the heat potentials (see [5], Ch. VI, Sec. 4), we find

$$0 = \lim_{x\to h(t)-0} V_x(t, x) = V_x(t, h(t)) - {}^1\!/_2 u_x(t, h(t)) \tag{4.2}$$

Let us consider the function $V(t, x)$ for $x \geqslant h(t)$, $0 \leqslant t \leqslant T$. Since $V(t, h(t)) = 0$, $V(0, x) = 0$ for $x > 0$ and $|V(t, x)| \leqslant 2c_3$, then $V(t, x) \equiv 0$ by the maximum principle, from which it results that

$$0 = \lim_{x\to h(t)+0} V_x(t, x) = V_x(t, h(t)) + {}^1\!/_2 u_x(t, h(t)) \tag{4.3}$$

Comparing (4.2) and (4.3), we conclude that $u_x|_{x=h(t)} \equiv 0$.

We obtain another functional equation by differentiating (2.1) with respect to x and equating the derivative $u_x(t, x)$ to zero for $x = h(t)$ by virtue of the second condition in (1.2):

$$-g(0)\frac{1}{\sqrt{t}}\exp\left(-\frac{h^2(t)}{4t}\right) =$$

$$= \frac{1}{4}\int_0^t (t-\tau)^{-5/2}\,[2(t-\tau) - h^2(t)]\,\exp\left(-\frac{h^2(t)}{4(t-\tau)}\right) f(\tau)\,d\tau +$$

$$+ \frac{1}{2}\int_0^t \frac{1}{\sqrt{t-\tau}}\left[\exp\left(-\frac{(h(t)+h(\tau))^2}{4(t-\tau)}\right) + \exp\left(-\frac{(h(t)-h(\tau))^2}{4(t-\tau)}\right)\right] g'(\tau)\,d\tau \tag{4.4}$$

Finally, we obtain the third functional equation from (2.1) by equating the functions $\partial u/\partial t$ and $du/dt = A(h)' \equiv g'(t)$ on the desired boundary $x = h(t)$ (here the property is used of the jump in the heat potential of a double layer under the assumption of the existence of a derivative $h'(\tau)$ which is integrable in absolute value with weight $(t-\tau)^{-1/2}$ in any half-interval $(0, t]$):

$$g'(t) = -g(0)\frac{h(t)}{t\sqrt{\pi t}}\exp\left(-\frac{h^2(t)}{4t}\right) +$$

$$+\frac{1}{\sqrt{\pi}}\frac{\partial}{\partial t}\int_0^t \frac{x}{(t-\tau)^{3/2}}\exp\left(-\frac{x^2}{4(t-\tau)}\right)f(\tau)\,d\tau\,|_{x=h(t)} -$$

$$-2\int_0^t G_\xi(h(t), h(\tau), t, \tau)\,g'(\tau)\,d\tau \qquad (4.5)$$

The proof of the equivalence of (4.4) and (4.5) to the problem (1.1), (1.2) is exactly the same as in the case of (4.1), by direct verification that the function $u(t, x)$, defined by (2.1), satisfies all the requirements of the definition of the solution of the problem (1.1), (1.2), but, it is true, under some additional a priori assumptions with respect to the solution of (4.4) and (4.5). For example, in the case of (4.5) it is sufficient to demand compliance with the property (2.5). Then, as is seen from (2.1), $u(t) \equiv u(t, h(t)) \to g(0)$ as $t \to 0$, and from the fact that $h(t)$ is a solution of (4.5), there results that the derivatives $u'(t)$ and $g'(t)$ coincide; therefore, $u(t, h(t)) \equiv g(t)$ and it is sufficient to carry out the same reasoning as for (4.1).

The obtained functional equations have substantially different properties and they should be applied depending on what information on the solution must be obtained. For example, in order to obtain the asymptotic $h(t)$ near $t = 0$ it is most convenient to use (4.1); for a clear operator formulation of the question of solvability of the problem (1.1), (1.2) it is most natural to use (4.5), which, as is easily seen from the proof of Theorem 2.1 and the derivation of (4.5), is a linear Volterra equation in $\partial u/\partial t \equiv g'(t)$ on the boundary $x = h(t)$, with $h(t)$ a given function, for the solution $u(t, x)$ of the following problem:

$$u_t = u_{xx} \qquad (4.6)$$

$$u\,|_{x=0} = f(t), \qquad u_x\,|_{x=h(t)} = 0, \qquad \lim_{t\to 0} u(t, h(t)) = \text{const} = g(0) \quad (4.7)$$

Let us turn to applications of the obtained results.

5. Asymptotic behavior of the unknown boundary $x = h(t)$ near $t = 0$.

Let us note first that a nonrigorously correct result may be obtained by starting from the equality $\partial u/\partial t = du/dt = g'(t)$ on the boundary $x = h(t)$ as $t \to 0$, which already contains the condition $u_x = 0$ for $x = h(t)$, by discarding the whole continuous (regular) part of the function $u(t, x)$ in the evaluation of $\partial u/\partial t$ (Theorem 2.1 and Corollary 2.1).

For a rigorous derivation of the asymptotic behavior we make some natural a priori assumptions on the "data" of the problem (1.1), (1.2) and the function $h(t)$.

We shall assume that

$$\operatorname{sign}\,[g(0) - f(0)] = -\operatorname{sign}\, g'(0) \neq 0 \qquad (5.1)$$

(from this latter it will be seen that the condition of agreement between the signs of the quantities $g(0) - f(0)$ and $-g'(0)$ is a necessary condition for the existence of a solution of the problem (1.1), (1.2)). For definiteness in the subsequent computations, we shall consider that

$$g(0) - f(0) > 0, \qquad g'(0) < 0 \qquad (5.2)$$

With respect to the function $h(t)$ we assume that for a sufficiently small $\delta_0 > 0$ this function is monotonous and convex upward on the segment $0 \leqslant t \leqslant \delta_0$, where the function $\omega(t) \equiv h(t)/2\sqrt{t}$ tends monotonously to $+\infty$ as $t \to 0$ and $|\omega'(t)| = o(t^{-1}\omega(t))$ (the last constraints are in complete agreement with condition (2.5); the assumption on $\omega'(t)$ becomes completely justified after the upper bound has been obtained for $\omega(t)$, where its derivation, as will be seen later, does not rely on this assumption; relative to this assumption, see also the Remark 5.1 below). Since the function $h'(t)$ may have only a countable set of discontinuous points in $[0, \delta_0]$, then without limiting the generality of the subsequent reasoning, it can be considered that $h'(t)$ exists everywhere in $[0, \delta_0]$.

By virtue of (5.2) it can be considered that $g'(t) < 0$ for $0 \leqslant t \leqslant \delta_0$.

Theorem 5.1. Under the above-mentioned assumptions

$$h(t) \sim 2\sqrt{t}\,\omega_0(t) \tag{5.3}$$

where $\omega_0(t)$ is a solution of Eq.(*)

$$[g(0) - f(0)]\,\omega_0(t)\exp(-\omega_0^2(t)) = -\sqrt{\pi}\,g'(0)\,t \tag{5.4}$$

Proof. Let us use (4.1). The first member on the right-hand side of this equation (we retain the notation J_f for it) is transformed by using the identity $f(\tau) = f(\tau) - f(0) + f(0)$:

$$J_f = [f(0) + o(1)]\int_0^t \frac{1}{2\sqrt{\pi}}\frac{h(t)}{(t-\tau)^{3/2}}\exp\left(-\frac{h^2(t)}{4(t-\tau)}\right)d\tau =$$

$$= [f(0) + o(1)]\left[1 - \Phi\left(\frac{h(t)}{2\sqrt{t}}\right)\right] \tag{5.5}$$

We separate the second member in the right-hand side of (4.1) into two terms $J^+ + J^-$, where

$$J^{\pm} = \frac{1}{2}\int_0^t \left[1 - \Phi\left(\frac{h(t) \pm h(\tau)}{2\sqrt{t-\tau}}\right)\right] g'(\tau)\,d\tau$$

Let us note that because of the monotonous behavior of $h(t)$:

$$0 \geqslant J^+ = \frac{1}{2}\int_0^t \left[1 - \Phi\left(\frac{h(t) + h(\tau)}{2\sqrt{t-\tau}}\right)\right] g'(\tau)\,d\tau \geqslant \tag{5.6}$$

$$\geqslant \frac{1}{2}\int_0^t \left[1 - \Phi\left(\frac{h(t)}{2\sqrt{t}}\right)\right] g'(\tau)\,d\tau = \frac{1}{2}\,t g'(0)\,\alpha_1(t)\,[1 - \Phi(\omega(t))]$$

where $\alpha_1(t) \to 1$ as $t \to 0$ (later we denote functions possessing this property by $\alpha_i(t)$).

Let us obtain lower and upper bounds for J^-. Using the monotonous behavior of $\omega(t)$, we have

$$J^- = \int_0^t \left[1 - \Phi\left(\frac{\sqrt{t}\,\omega(t) - \sqrt{\tau}\,\omega(\tau)}{\sqrt{t-\tau}}\right)\right] g'(\tau)\,d\tau \leqslant$$

$$\leqslant g'(0)\,\alpha_2(t)\int_0^t \left[1 - \Phi\left(\frac{\sqrt{t} - \sqrt{\tau}}{\sqrt{t-\tau}}\,\omega(t)\right)\right] d\tau$$

Making the substitution

$$\theta = \frac{\sqrt{t} - \sqrt{\tau}}{\sqrt{t-\tau}}\,\omega(t)$$

we have, furthermore

$$J^- \leqslant g'(0)\,\alpha_2(t)\,4t\omega^{-2}\int_0^{\omega} [1 - \Phi(\theta)]\,\theta\,\frac{\omega^4(\omega^2 - \theta^2)}{(\omega^2 + \theta^2)^3}\,d\theta =$$

$$= 4g'(0)\,\alpha_3(t)\,\omega^{-2}\int_0^{+\infty} [1 - \Phi(\theta)]\,\theta\,d\theta$$

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*) It is easy to see that $\omega_0(t) \sim \sqrt{\ln(1/t)}$ for $t \to 0$; however, the function $2\sqrt{t}\,\omega_0(t)$ characterizes the asymptotic $h(t)$ more accurately.

By integration by parts we express the last integral in terms of $\Phi(+\infty) = 1$ and equal to ¼. Thus

$$J^- \leqslant \alpha_3(t)\ g'(0)t\ \omega^{-2}(t) \tag{5.7}$$

To derive the lower bound, let us note that by virtue of the convexity of $h(t)$:

$$\frac{h(t)-h(\tau)}{2\sqrt{t-\tau}} \geqslant \frac{h'(t)}{2}\sqrt{t-\tau}$$

and therefore

$$J^- \geqslant \frac{1}{2}g'(0)\,\alpha_4(t)\int_0^t\left[1-\Phi\left(\frac{h'(t)}{2}\sqrt{t-\tau}\right)\right]d\tau = 4g'(0)\,\alpha_4(t)\frac{1}{(h'(t))^2}\int_0^{\chi(t)}[1-\Phi(\sigma)]\sigma\,d\sigma$$

$$\sigma = h'(t)\sqrt{t-\tau}/2, \qquad \chi(t) = {}^1\!/_2 h'(t)\sqrt{t}$$

Since

$$h'(t) = \frac{1}{\sqrt{t}}\omega(t) + 2\sqrt{t}\omega'(t) \sim \frac{1}{\sqrt{t}}\omega(t), \quad 2\chi(t) = h'(t)\sqrt{t} > \frac{1}{2}\omega(t) \to \infty$$

by virtue of the condition $|\omega'(t)| = o(t^{-1}\omega(t))$ as $t \to 0$, then

$$J^- \geqslant 4g'(0)\,\alpha_5(t)\frac{1}{(h'(t))^2}\int_0^\infty[1-\Phi(\sigma)]\,\sigma\,d\sigma \geqslant g'(0)\,t\alpha_6(t)\,\omega^{-2}(t) \tag{5.8}$$

From the estimates (5.6) to (5.8) there results that

$$J^+ + J^- = g'(0)\,t\,\alpha_7(t)\omega^{-2}(t) + o(1)\ \ [1-\Phi(\omega(t))] \tag{5.9}$$

Substituting (5.5) and (5.9) into (4.1), we have

$$[g(0) - f(0) + o(1)]\ \ [1-\Phi(\omega(t))] = -\alpha_7(t)g'(0)t\ \omega^{-2}(t)$$

Taking into account that as $\omega \to +\infty$:

$$1-\Phi(\omega) \sim \exp(-\omega^2)/\sqrt{\pi}\omega$$

we obtain the confirmation of Theorem 5.1(*)

Remark 5.1. The condition $|\omega'(t)| = o(t^{-1}\omega(t))$ is not essential to the Proof of the Theorem since the differential inequality

$$[g(0) - f(0)]\ \omega^{-1}\ \exp(-\omega^2) \leqslant -g'(0)\ \ \alpha_8\ (t)\ \sqrt{\pi}/(h'(t))^2$$

proved above may be used to obtain the lower bound for $\omega(t)$, however, this requires some additional considerations.

6. Problem of impact of a viscoplastic rod on a rigid obstacle.

This problem (see [1]) is a particular case of (1.1), (1.2) for

$$f(t) \equiv 0, \qquad A(h) = 1 - s\int_0^t \frac{d\tau}{1-h(\tau)} \equiv g(t), \qquad s = \mathrm{const} > 0$$

where it is natural to take a manifold of continuous functions $h(t)$, $h(0) = 0$, $0 < h(t) < 1$ for $0 < t \leqslant T$ as the set of admissible boundaries (the length of the rod is unity in nondimensional variables). It is clear from the physical formulation of this problem that its solution exists only in the small: $h(T) = 0$ and $u(T, 0) = 0$ for some $T > 0$.

Compliance with all the assumptions made in the preceding Sections relative to the data of problem (1.1), (1.2), except property (2) of the operator A, is perfectly evident here. Let us verify this property.

Let $h_1(t)$ and $h_2(t)$ be some functions from the set of admissible boundaries, where $h_1(t_0) = h_2(t_0)$ (therefore, both these functions are defined and continuous in $[0, t_0]$). Let t^* denote the upper bound of those $t < t_0$, for which $h_1(t) = h_2(t)$; evidently the set of such t is not empty (since it contains $t = 0$), and that $h_1(t^*) = h_2(t^*)$. By the definition of t^* the inequality

*) The following property of the inverse $\omega(z)$ is taken into account for $z = \omega e^{-\omega^2}$: $\omega(kz) \sim \omega(z)$ as $z \to 0$ for any $k > 0$.

$$h_1(t) > h_2(t) \tag{6.1}$$

is satisfied for $t^* < t \leqslant t_0$

Taking into account that $0 \leqslant h_i(t) < 1$ and using (6.1), we have

$$g^{h_1}(t^*) - g^{h_1}(t_0) = s\int_{t^*}^{t_0}\frac{d\tau}{1-h_1(\tau)} > s\int_{t^*}^{t_0}\frac{d\tau}{1-h_2(\tau)} = g^{h_2}(t^*) - g^{h_2}(t_0)$$

q.e.d. Hence, the considered problem may have just one solution.

Let us show that the solution of the problem of rod impact may not exist infinitely long. In fact, if the solution $u(t, x)$, $h(t)$ of this problem is defined in the domain $\{0 \leqslant x \leqslant h(t),\ 0 < t < +\infty\}$, then because of the obvious inequality $u(t, h(t)) < 1 - st$ the function $u(t, h(t)) < 0$ for $t > 1/s$, which contradicts the maximum principle (Corollary 2.3), according to which $0 \leqslant u(t, x) \leqslant 1$ (we do not consider the physically unreal case $h(t_1) = 1$, $t_1 > 0$ (see [1], say)). From this reasoning there also results the upper bound for the segment $[0, T]$ of existence of the solution

$$T \leqslant 1/s$$

In this case the functional Eq. (4.1) is

$$1 - \Phi\left(\frac{h(t)}{2\sqrt{t}}\right) = \frac{s}{2}\int_0^t\left[2 - \Phi\left(\frac{h(t)+h(\tau)}{2\sqrt{t-\tau}}\right) - \Phi\left(\frac{h(t)-h(\tau)}{2\sqrt{t-\tau}}\right)\right]\frac{d\tau}{1-h(\tau)} \tag{6.2}$$

Near $t = 0$ the asymptotic $h(t)$ has the form

$$h(t) \sim 2\sqrt{t}\,\omega_0(t), \quad \omega_0(t)\exp(-\omega_0^2(t)) = s\sqrt{\pi}t, \quad \omega_0 \sim \sqrt{-\ln t}$$

Let us obtain the lower bound for $h(t)$ on the whole segment of existence of the solution. Taking into account that

$$\Phi\left(\frac{h(t)+h(\tau)}{2\sqrt{t-\tau}}\right) + \Phi\left(\frac{h(t)-h(\tau)}{2\sqrt{t-\tau}}\right) \geqslant 0$$

for any nonnegative $h(t)$ and $h(\tau)$, we have from (6.2)

$$1 - \Phi\left(\frac{h(t)}{2\sqrt{t}}\right) \leqslant s\int_0^t \frac{d\tau}{1-h(\tau)} \tag{6.3}$$

There hence results that $h(t) \geqslant 2\sqrt{t}\,\omega_-(t)$, where $\omega_-(t)$ is a solution of Eq.

$$1 - \Phi(\omega_-(t)) = s\int_0^t \frac{d\tau}{1 - 2\sqrt{\tau}\,\omega_-(\tau)}$$

or the corresponding differential Eq.

$$\omega' = -s\sqrt{\pi}\,\frac{\exp(\omega^2)}{1 - 2\sqrt{t}\,\omega}, \qquad \omega(0) = +\infty \tag{6.4}$$

It is easy to see that the single unbounded solution, as $t \to 0$, of the problem (6.4) is the separatrix of this equation, which divides smooth solutions of this equation from solutions having vertical tangents. It is clear that if $\omega(t)$ is some solution of (6.4), which intersects the t-axis at the point $t = T_0$, then the bound $T > T_0$ is valid for T, and any solution passing through the ω axis will intersect the t-axis (it is easy to show that the separatrix also intersects the t-axis).

As has been remarked above, a point T exists where $h(T) = 0$. Information on the nature of the behavior of $h(t)$ near $t = T$ may be obtained directly from the formulation of the problem. To do this we prove the following assertion which is of independent interest.

L e m m a. Let $u(t, x)$ be the solution of (1.1) in the domain

$$Q\{0 < x < \varphi(T - t),\ T_0 < t < T;\ \varphi(0) = 0,\ \varphi \in C[T_0 - T,\ 0]\}$$

which is continuous in $\overline{Q}$ and satisfies the conditions

$$u|_{x=0} = u_x|_{x=\varphi(T-t)} = 0$$

Let $\varphi(\sigma) = o(\sqrt{\sigma})$ as $\sigma \to 0$. Then $u(t, x) \to 0$ as $t \to T$ more rapidly than any power of $(T - t)$.

Proof. Let us make the change of variable

$$e^{-\tau} = T - t, \quad y = x / \sqrt{T-t}$$

The domain Q is hence transformed into the domain

$$D\{(\tau, y): 0 < y < e^{\tau/2}\varphi(e^{-\tau}), \qquad -\ln(T - T_0) \leqslant \tau < +\infty\}$$

and (1.1) goes over into

$$L(u) \equiv u_{yy} - {}^1\!/_2\, y u_y - u_\tau = 0$$

Let $D_0 = D \cap \{(\tau, y): \tau \geqslant \tau_0\}$. Let us choose an arbitrary $\alpha > 0$, and let us consider the function

$$v(y) = 2 - e^{-\alpha y} < 2$$

in D_0.

Since for sufficiently large τ_0 the domain D_0 is contained because of the condition $\varphi(\sigma) = o(\sqrt{\sigma})$ in some half-strip $\{0 \leqslant y \leqslant e^{\tau_0/2}\varphi(e^{-\tau_0}),\ \tau \geqslant \tau_0\}$, as narrow as desired, then for given α a τ_0 exists in D_0 such that:

$$L(v) \equiv (-\alpha^2 - {}^1\!/_2\, y\, \alpha)e^{-\alpha y} \leqslant -{}^1\!/_2\, \alpha^2$$

Evidently $v(y) > 1$ for any $\alpha > 0$, and $v_y = \alpha e^{-\alpha y} > 0$ on the right-hand side boundary Γ of the domain D_0. Let us put $w \equiv u/v$. It is clear that

$$w|_{y=0} = 0, \ w_y|_\Gamma = -\frac{u}{v^2} v_y|_\Gamma \equiv -a(y) w|_\Gamma, \qquad a = \frac{v_y}{v}\bigg|_\Gamma > 0$$

In D_0 the function $w(\tau, y)$ satisfies Eq.

$$w_{yy} + \left(2\frac{v_y}{v} - \frac{1}{2} y\right) w_y + \frac{L(v)}{v} w - w_\tau = 0$$

where $(L(v) / v) \leqslant -\alpha^2 / 4$. Applying the maximum principle to the functions

$$w^\pm \equiv \pm w - M_0 \exp[(\alpha^2/4)(\tau_0 - \tau)], \text{ where } M_0 = \max_y |w(\tau_0, y)|$$

we obtain the estimate

$$|w(\tau, y)| \leqslant M_0 e^{(\alpha^2/4)(\tau_0 - \tau)}$$

therefore

$$|u(\tau, y)| \leqslant M(\alpha) e^{-\alpha^2\tau/4}$$

Hence, the assertion of the Lemma results by virtue of the arbitrariness of $\alpha > 0$.

Since $u(t, h(t)) - u(T, h(T)) \sim s(T - t)$ for any $h(t)$ in the problem of rod impact, the order of the contact of the function $h(t)$ to the line $t = T$ may not be less than for some second degree parabola. It turns out that the order of contact of $h(t)$ to this line is stronger than for any second degree parabola. In fact, making the very same replacement in the opposite case as in the Lemma, and applying an eigenfunction expansion of the problem

$$-Y'' + {}^1\!/_2\, y\, Y' = \lambda Y, \qquad Y(0) = Y'(l) = 0$$

to $u(\tau, y)$ in the new variables for some $l > 0$, we arrive at a contradiction to the fact that, by virtue of the condition

$$u(t, h(t)) = 1 - s\int_0^t \frac{d\tau}{1 - h(\tau)}$$

this problem should have the eigenvalues $\lambda = n/2$, $n = 2, 3, 4, \ldots$, which cannot be for any l, as is easily seen. (For $\lambda = n/2$ the general solution of the considered ordinary equation is expressed in terms of polynomials and Chebyshev-Hermite functions).

Therefore, for the problem of rod impact (for some regularity conditions on the function $t = h^{-1}(x) \sim T$).

$$\overline{\lim_{t \to T}} \frac{h(T - t)}{\sqrt{T - t}} = +\infty$$

7. Some remarks and particularly extensions.

Remark 7.1. More general problems may be considered analogously; problems with initial conditions when $h(0) = l > 0$, and $u(0, x) = u_0(x)$ for $0 \leqslant x \leqslant l$, $u_0(l) = g(0)$; it is easy to indicate the conditions on $u_0(x)$ for which for example the uniqueness theorem would remain valid. However, for the existence theorem to be valid some conditions on the consistency between $u_0(x)$ and the operator A are certainly necessary. For example, as has been shown in [6], the formulation of the rod impact problem will be contradicted if $l > 0$ and $u_0(x) \equiv 1$ (it is enough to compare the order of smallness of $1 - u(t, x_0)$, $x_0 < l$ and $1 - u(t, h(t))$ as $t \to 0$ by using the conditions $u_x \geqslant 0$, $u(t, x) \leqslant 1$).

Remark 7.2. The results obtained may be extended to the case of some parabolic equations with variable coefficients.

Remark 7.3. Questions of reducing problems with unknown boundaries to functional equations are considered in [7].

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